MATHEMATICAL MOSAICS ISLAMIC ART AND QUASICRYSTALS

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Talk given at

Crystallography for the next generation: the legacy of IYCr Hassan II Academy of Science and Technology, Rabat, Morocco 23 April 2015

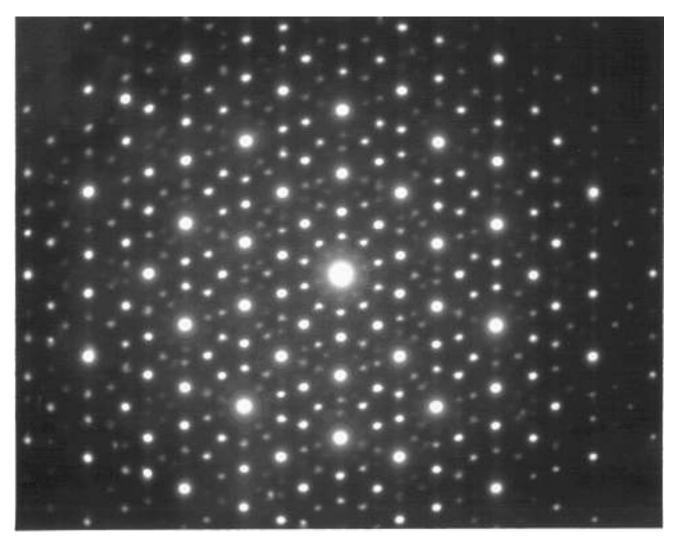


Boys washing up at communal water source, Sidi Ghalem.

Mathematical Mosaics, Islamic Art and Quasicrystals

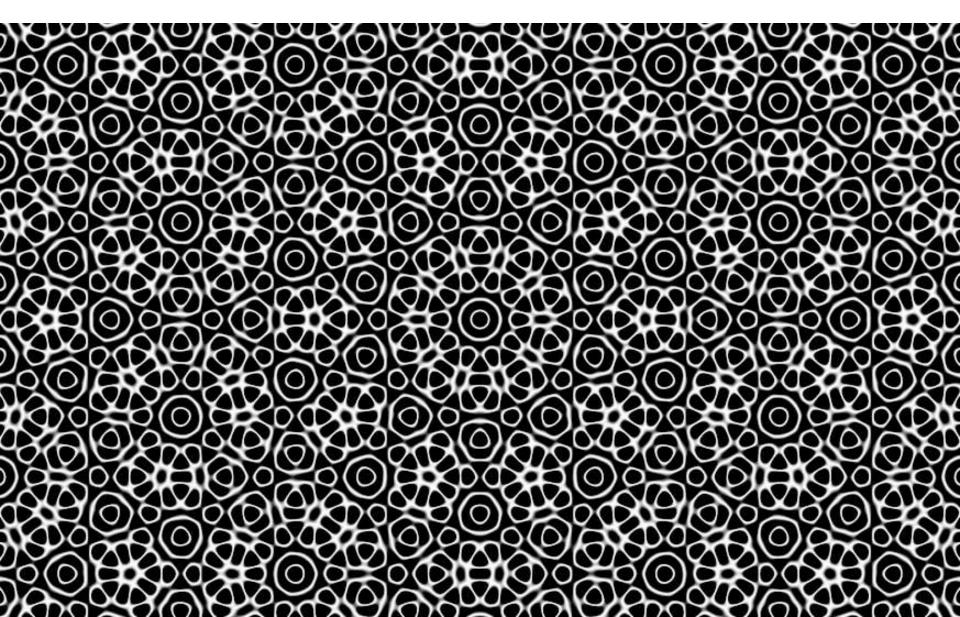
The use of local fivefold symmetry in Islamic tilings started in the 11th century. Examples of such tilings are discussed and the rules employed in these tilings are presented. It is demonstrated how a periodic pattern can be transformed into a locally aperiodic Penrose tiling pattern which most efficiently solves the difficult problem of the mathematical aperiodic tiling. Most remarkably, these tilings match with crystal diffraction patterns which exhibit fivefold symmetry. Inevitably, this topic reminds us of the flowering of scientific brilliance in Islam from the ninth to the thirteenth century—a period often described as the "golden age" of Islamic science.

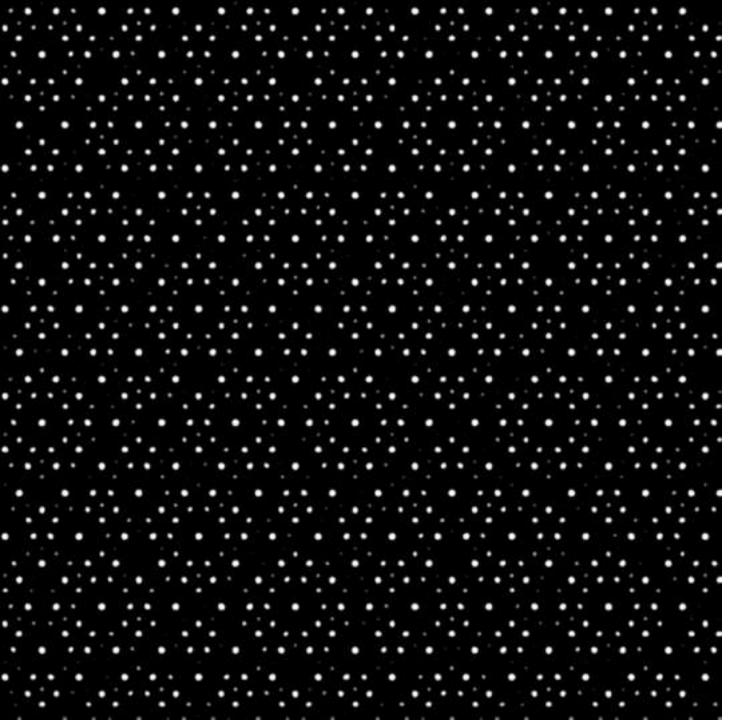
This is a diffraction pattern of a real 3D quasicrystal. For a 3D crystal, the diffraction pattern is also three dimensional. All of the rendered images here are for 2D quasicrystals with 2D diffraction patterns.



http://spacecollective.org/michaelerule/5810/Quasicrystal-Diffraction-Patterns

This is a 5-fold 2D quasicrystal spatial domain. This pattern is related to Penrose tiling



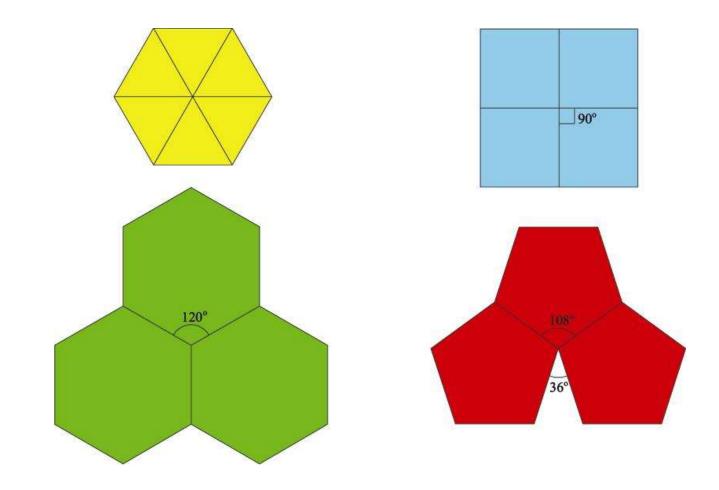


http://spacecollective.org/micha elerule/5810/Quasicrystal-Diffraction-Patterns

Five plane waves are summed and thresholded to generate this pattern. The Fourier transform of this lattice looks more like what one might expect for a crystal.

Joint work with Mustafa Sancak

- "New sets of pentaplex tiles" *Balkan Physics Letters* **11** (2003) 215.
- "Pentapleks Kaplamalalar" *Tübitak Popüler Bilim Kitapları* 254 (2007). (in Turkish)
- "Turkish-Islamic art and Penrose tilings" *Balkan Physics Letters* **15** (2007) 84 .



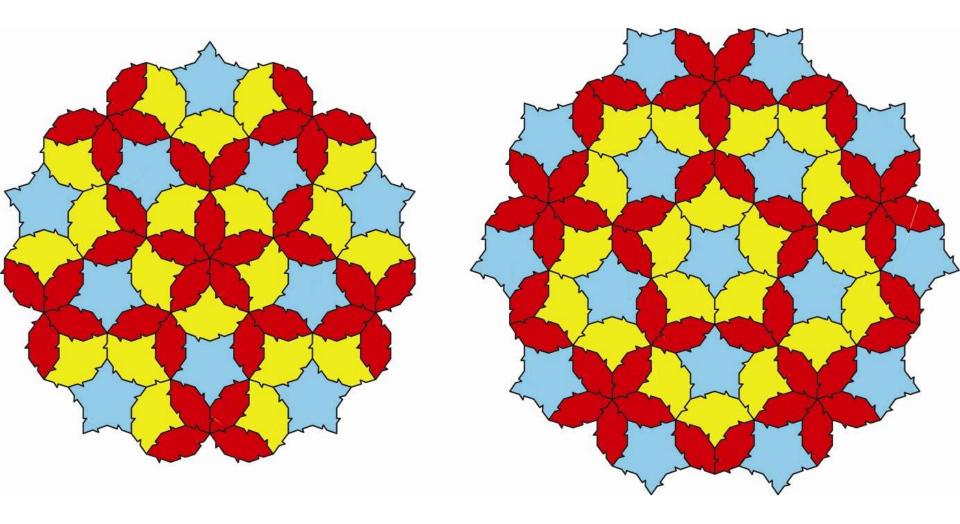
Mathematically speaking "Tiling of the plane" means "Infinite tiling"

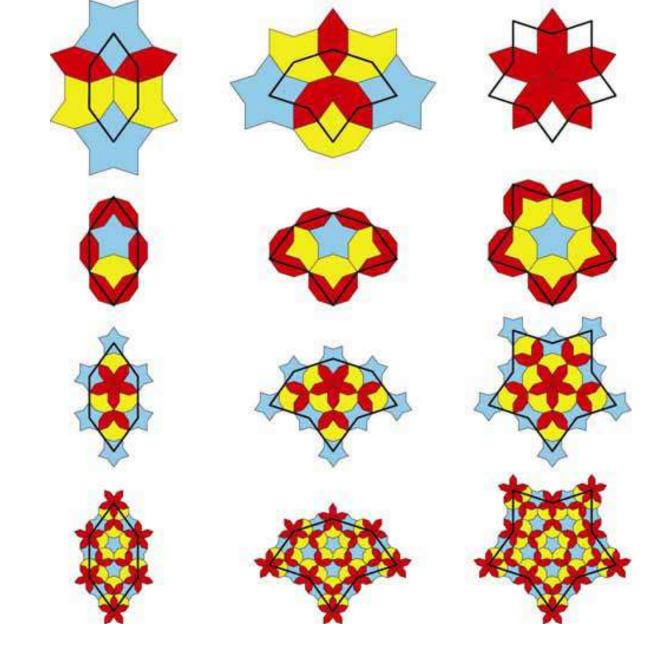
In art "Tiling" means "Finite tiling"

How to identify the 17 wallpaper groups for periodic tilings

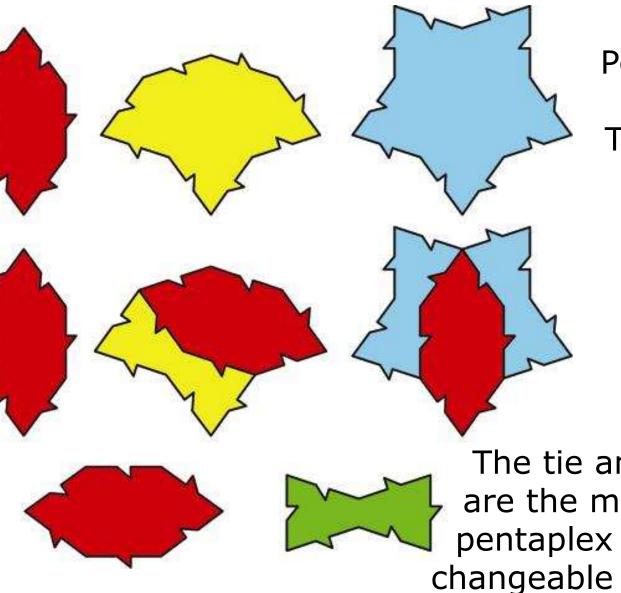
http://en.wikipedia.org/wiki/Wallpaper_group#Symmetries_of_patterns

Least	Has reflection?	
rotation	Yes	No
360° / 6	<u>p6m</u>	<u>p6</u>
360° / 4	Has mirrors at 45°? Yes: <u><i>p4m</i></u> No: <u><i>p4g</i></u>	<u>p4</u>
360° / 3	Has rot. centre off mirrors? Yes: <u>p31m</u> No: <u>p3m1</u>	<u>p3</u>
360° / 2	Has perpendicular reflections?YesNoHas rot. centre off mirrors? pmg Yes: cmm No: pmm	Has glide reflection? Yes: <u>pgg</u> No: <u>p2</u>
none	Has glide axis off mirrors? Yes: <u>cm</u> No: <u>pm</u>	Has glide reflection? Yes: <u>pg</u> No: <u>p1</u>



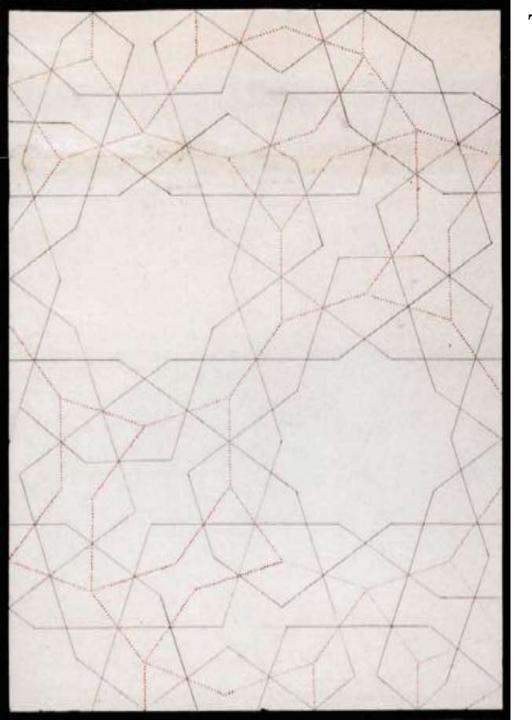


The method of inflation is used to enlarge a tiling

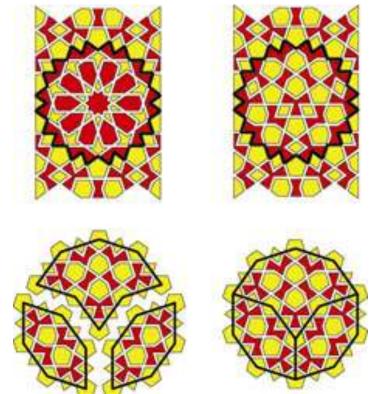


Pentaplex Map Tiles: Tie, Fish, Star

The tie and the bow are the map tiles for pentaplex tilings with changeable inner regions



The bow, the tie and the decagon in the Topkapi Scroll Panel 52



The decagon when filled with the bow and the tie loses its ten-fold rotational symmetry

http://en.wikipedia.org/wiki/Penrose_tiling

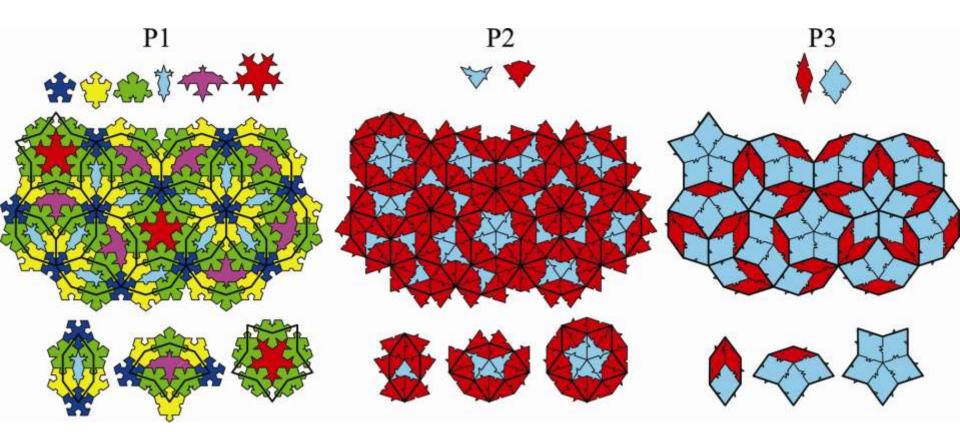
A Penrose tiling is a nonperiodic tiling generated by an aperiodic set of prototiles named after Roger Penrose, who investigated these sets in the 1970's. All tilings obtained with the Penrose tiles being non periodic, Penrose tilings are commonly, but not correctly, described as aperiodic tilings. Among the infinitely many possible tilings there are two that possess both mirror symmetry and fivefold rotational symmetry. A Penrose tiling has many remarkable properties, most notably:

- it is nonperiodic which means that it lacks any translational symmetry. More informally, a shifted copy will never match the original exactly.
- any finite region in a tiling appears infinitely many times in that tiling and, in fact, in any other tiling. This property would be expected if the tilings had translational symmetry so it is a surprising fact given their lack of translational symmetry.
- it is a quasicrystal: implemented as a physical structure a Penrose tiling will produce Bragg diffraction; the diffractogram reveals both the underlying fivefold symmetry and the long range order. This order reflects the fact that the tilings are organized, not through translational symmetry, but rather through a process sometimes called "deflation" or "inflation."

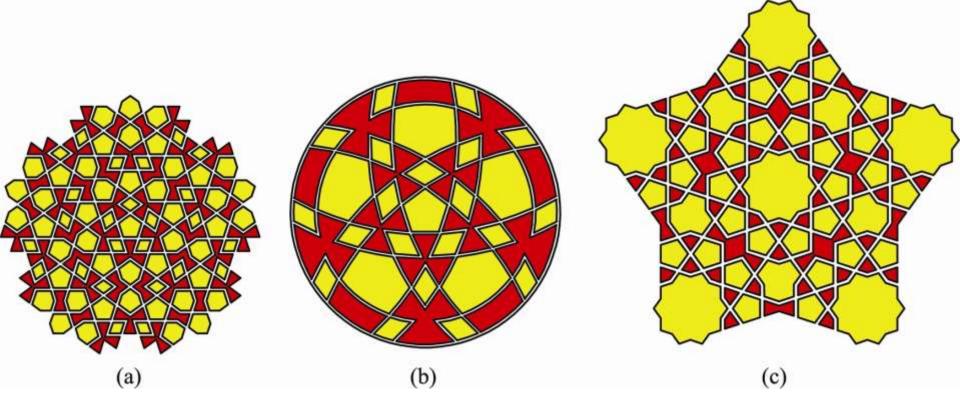
Historical background

The sets of tiles proposed by Penrose are among the most simple examples of a counterintuitive mathematical fact - the existence of aperiodic sets! In 1961, Hao Wang noted connections between problems in geometry— specifically problems about tiling— and a certain class of decision problem. As an aside, he observed that if the so-called Domino Problem were undecidable, then there would have to exist an aperiodic set of tiles. As the existence of such a set seemed implausible, Wang conjectured no such set could exist, and that the Domino Problem is decidable for tiles in the plane. In his 1964 thesis, Robert Berger disproved Wang's conjecture, proving that the Domino Problem is in fact undecidable, and producing an aperiodic set of 104 distinct tiles. (In his published monograph, Berger gives only a larger set of 20426 tiles.) This number was further reduced by Donald Knuth, and then Raphael Robinson, who gave an aperiodic set of just six tiles (and simplified Berger's proof) in an elegant 1971 paper. In 1972, Roger Penrose gave the first of several variations of tiles forcing a hierarchical pentagonal structure, a set of six tiles. Over the next several years, other variations were found, with the participation of Raphael Robinson, Robert Ammann and John H. Conway.

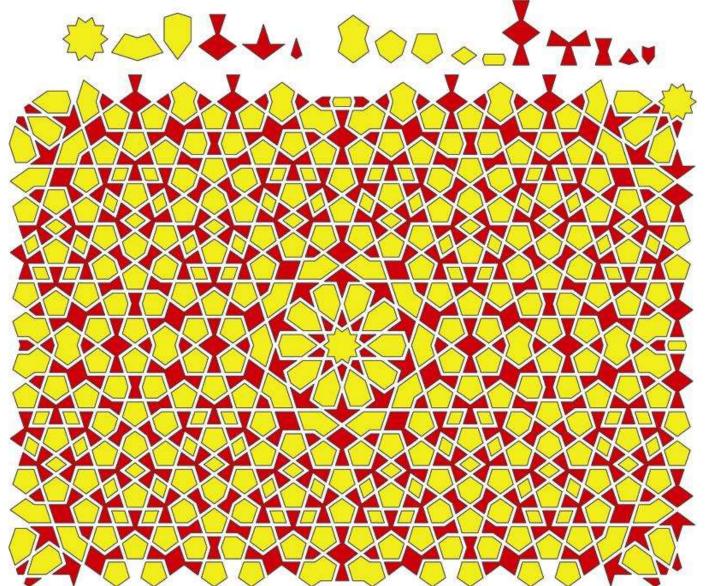
In 1981 De Bruijn explained a method to construct Penrose tilings from five families of parallel lines as well as a "cut and project method," in which Penrose tilings are obtained as two-dimensional projections from a five-dimensional cubic structure. In this approach the Penrose tiling is considered as a set of points, its vertices, while its tiles are just the geometrical shapes defined by connecting edges.



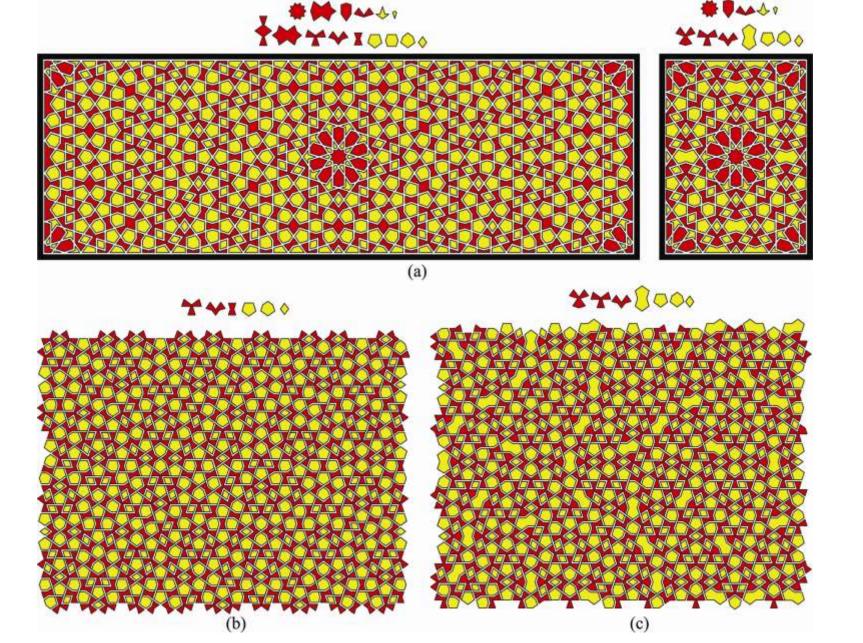
The three sets of pentaplex tiles P1, P2, P3 discovered by Penrose in 1973 and examples of aperiodic tilings using these sets. The three tilings can be converted into each other by using the map tiles, the tie, the fish and the star which are filled with the Penrose tiles as shown below each tiling. In terms of the tie, the fish, and the star, the three tilings are identical.



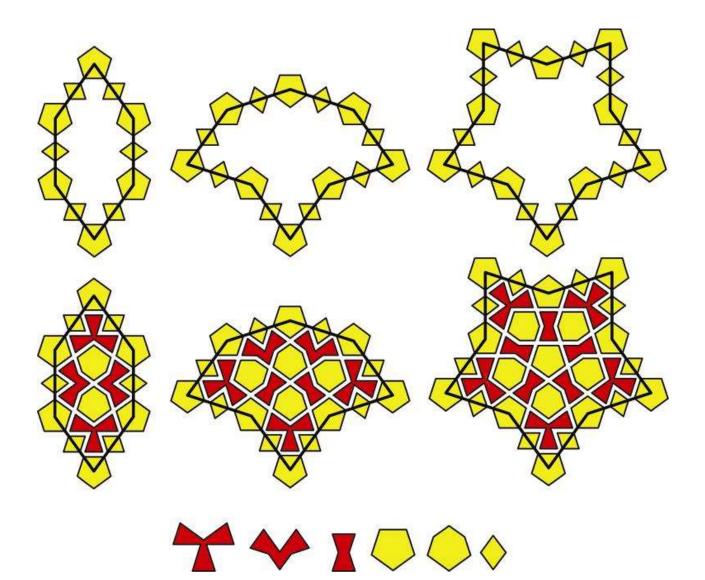
Three tilings with pentagonal symmetry. A modern example constructed by us (a). This tiling can uniquely be converted to a Penrose tiling. Two examples from the Seljuks: Dome of Friday Mosque, Isfahan (1089), (2) (b). Mosaics from Izzettin Keykavus Mausoleum, Sivas (1220), (3) (c).

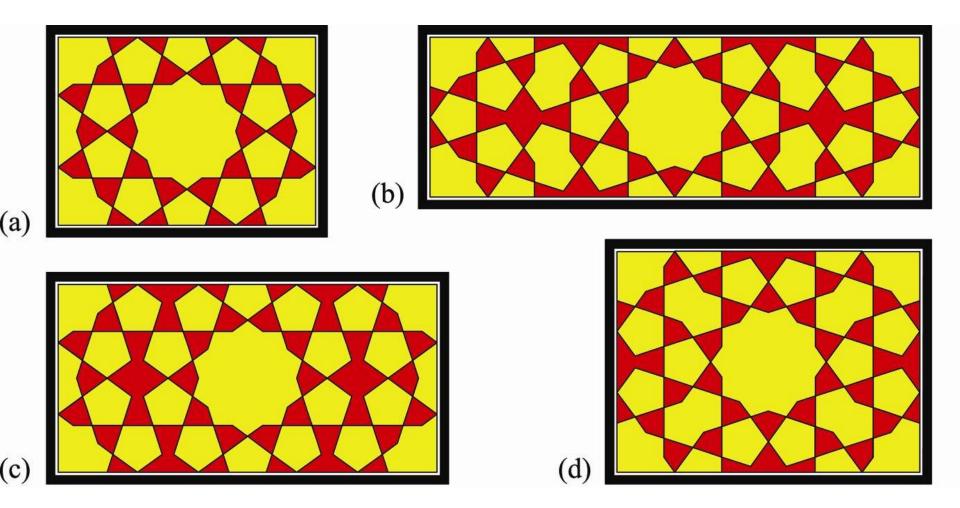


An Islamic tiling which uses pentagons and ten pointed stars. The number of tiles used is too large for the model to be considered mathematically interesting. Konya, Karatay Madrasa (1250).



Tilings on side doors of Suleymaniye Mosque, Istanbul (1550-1557) (a). Using six of these tiles (b) or seven of these tiles (c) one can construct aperiodic pentaplex tilings which can uniquely be converted into Penrose tilings.





Four examples of Islamic tiling patterns where a ten pointed star is placed at the center and one quarter of the star is placed at each corner of a rectangle. Karatay Mosque in Antalya (1250) (a), Yesil Mosque in Bursa (1424) (b), Abbasid Al-Mustansiriyya Madrasa in Baghdad (1227) (c), Beyazit Mosque in Istanbul (1506) (d).

Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture

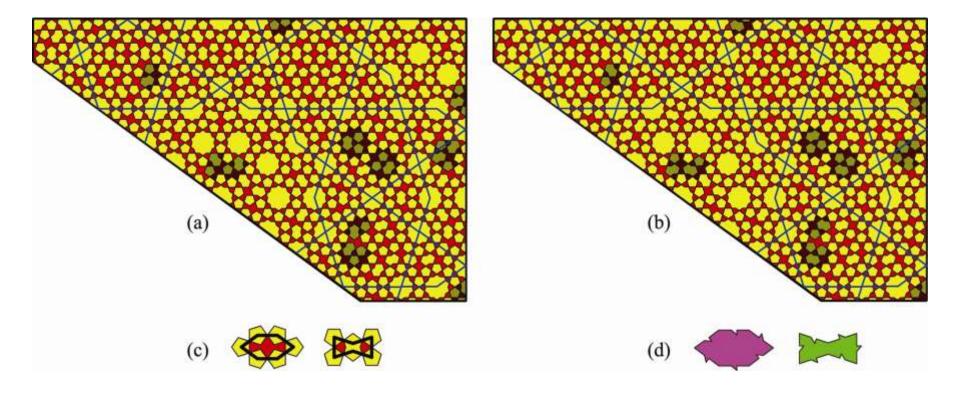
Peter J. Lu and Paul J. Steinhardt

The conventional view holds that girih (geometric star-and-polygon, or strapwork) patterns in medieval Islamic architecture were conceived by their designers as a network of zigzagging lines, where the lines were drafted directly with a straightedge and a compass. We show that by 1200 C.E. a conceptual breakthrough occurred in which girih patterns were reconceived as tessellations of a special set of equilateral polygons ("girih tiles") decorated with lines. These tiles enabled the creation of increasingly complex periodic girih patterns, and by the 15th century, the tessellation approach was combined with self-similar transformations to construct nearly perfect quasi-crystalline Penrose patterns, five centuries before their discovery in the West.

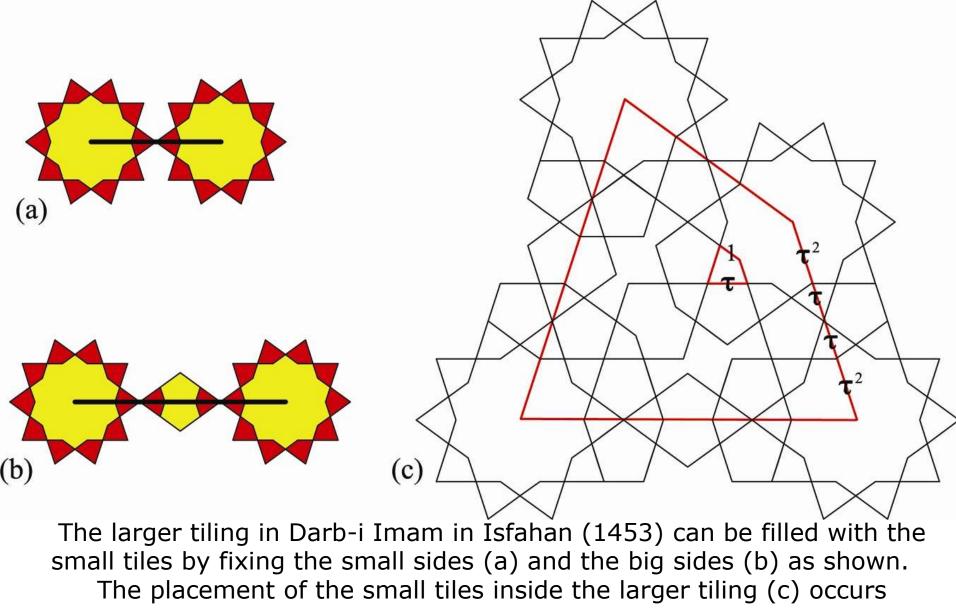
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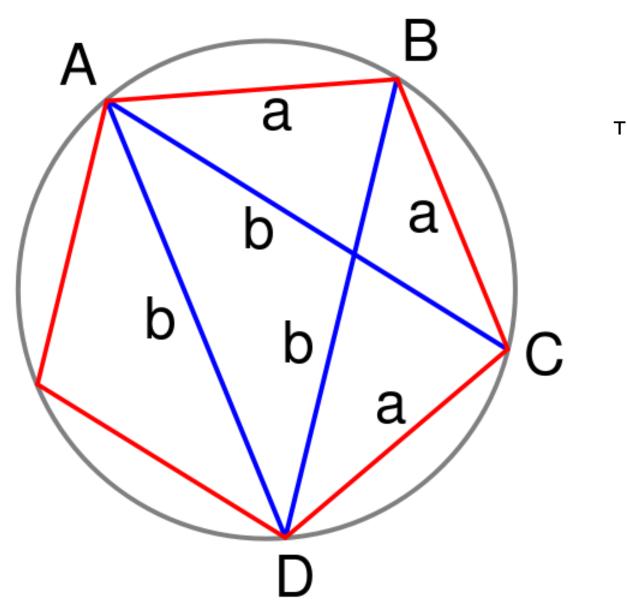
Darb-1 Imam, Isfahan, 1453



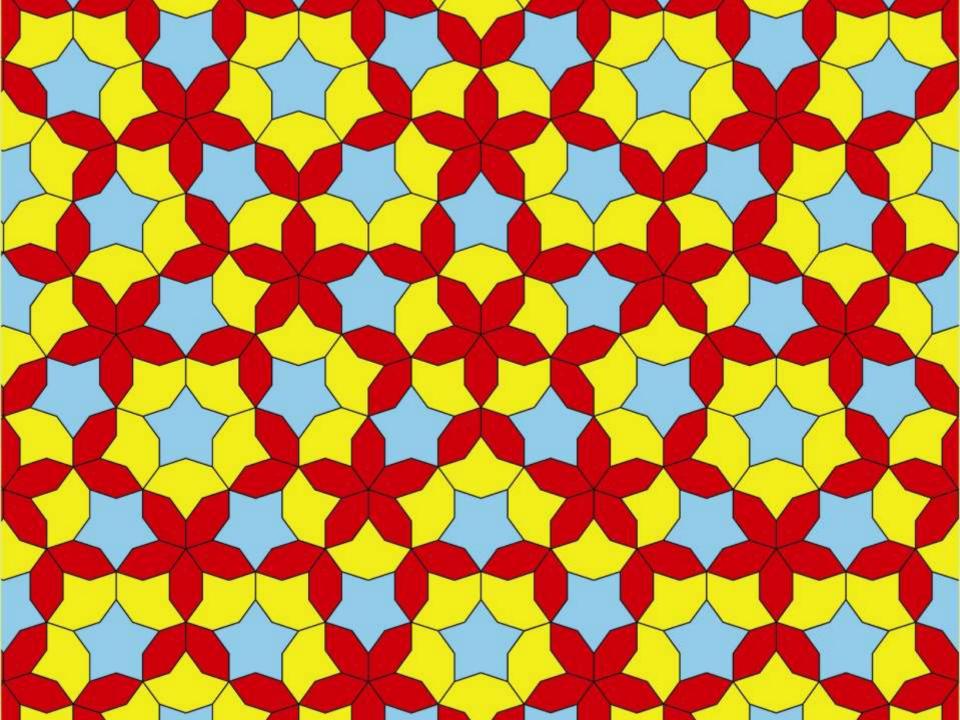
Comparison of the Darb-i Imam geometrical decorations (a) with the best fitting aperiodic pentaplex tiling (b). The shaded regions indicate violations of aperiodic pentaplex tiling. For this comparison the tiles have been converted to tie and bow tiles as shown in (c) which are then adorned with notches (d) so that they can cover the plane only aperiodically.

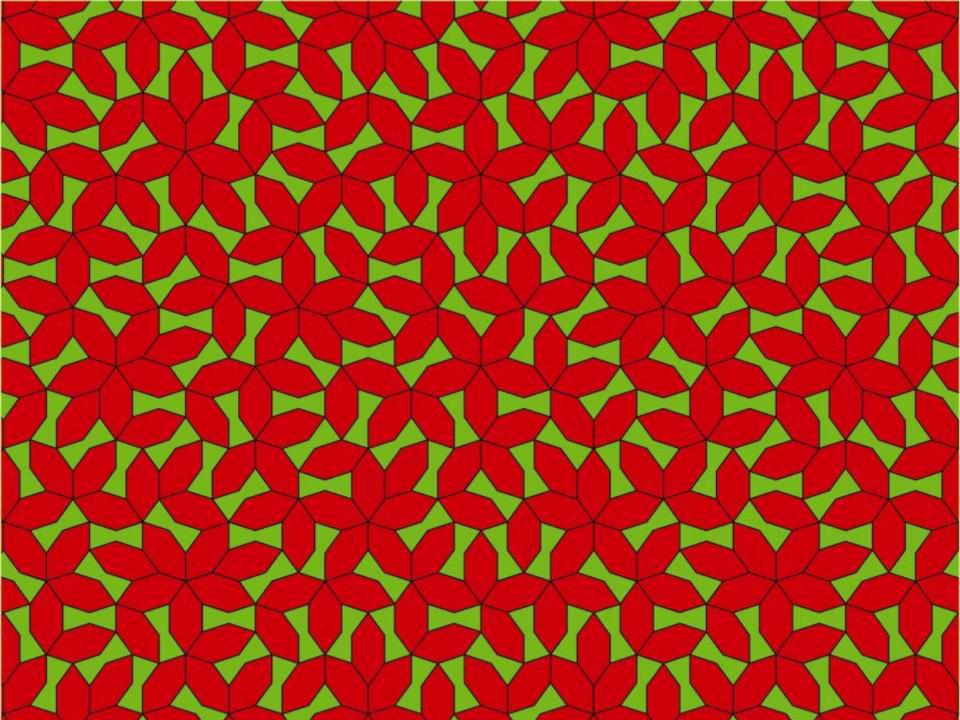


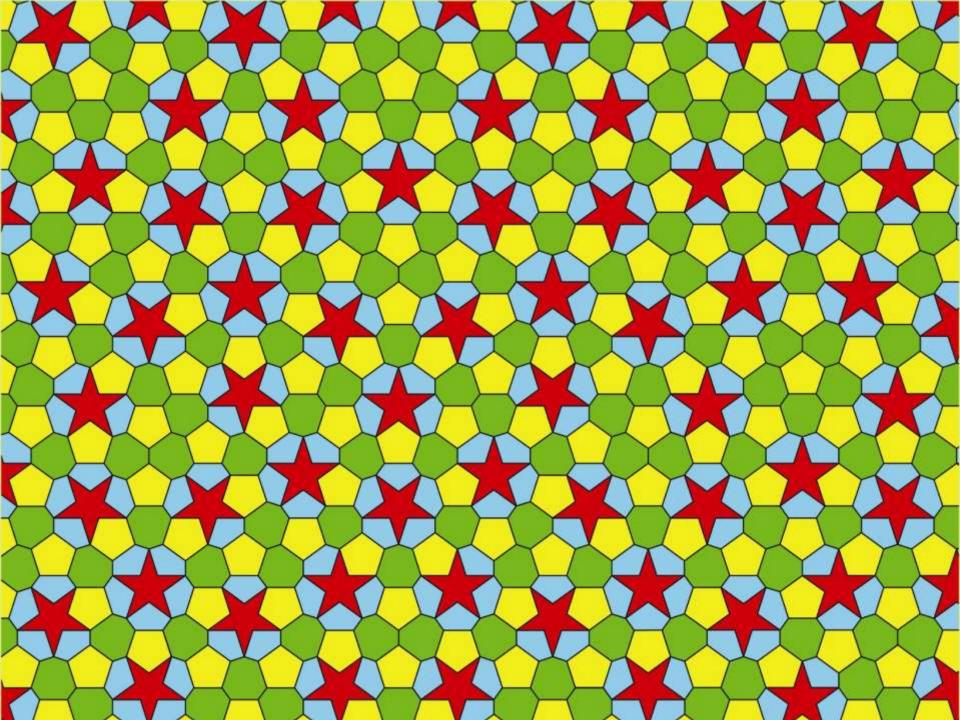
with a ratio of $2(\tau^2 + \tau) = 2\tau^3$ where τ is the golden ratio $(\tau^2 = \tau + 1)$. Darb-i Imam tiling is a portion of the pattern in Karatay Mosque in Antalya (1250) which perfectly matches the black lines whose insides have been filled with smaller tiles.

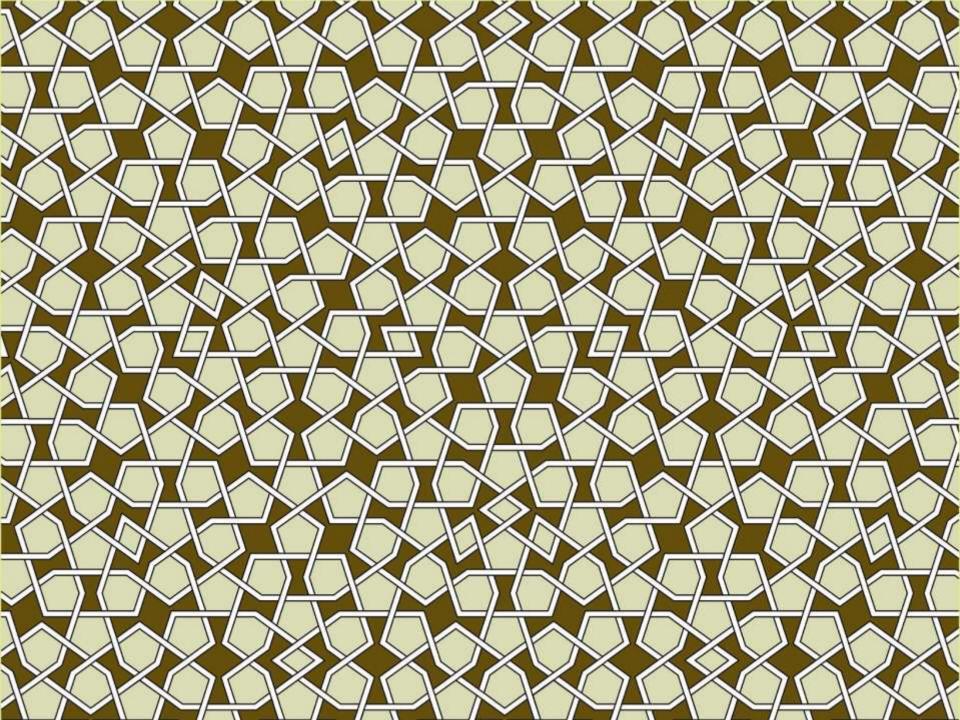


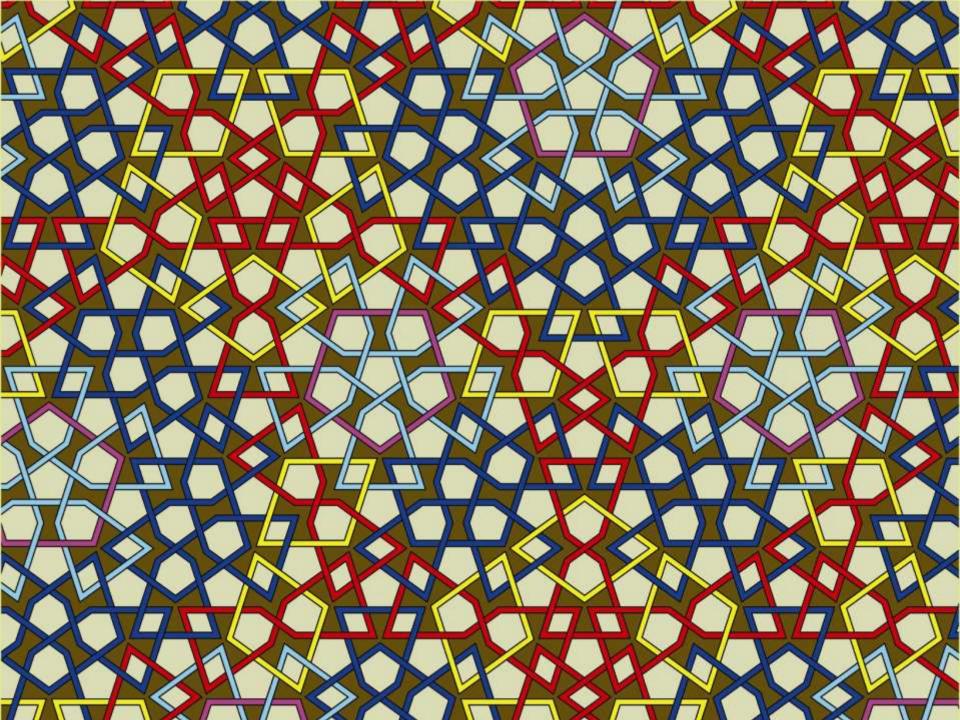
$$r = \frac{b}{a} = \frac{(1+\sqrt{5})}{2}.$$

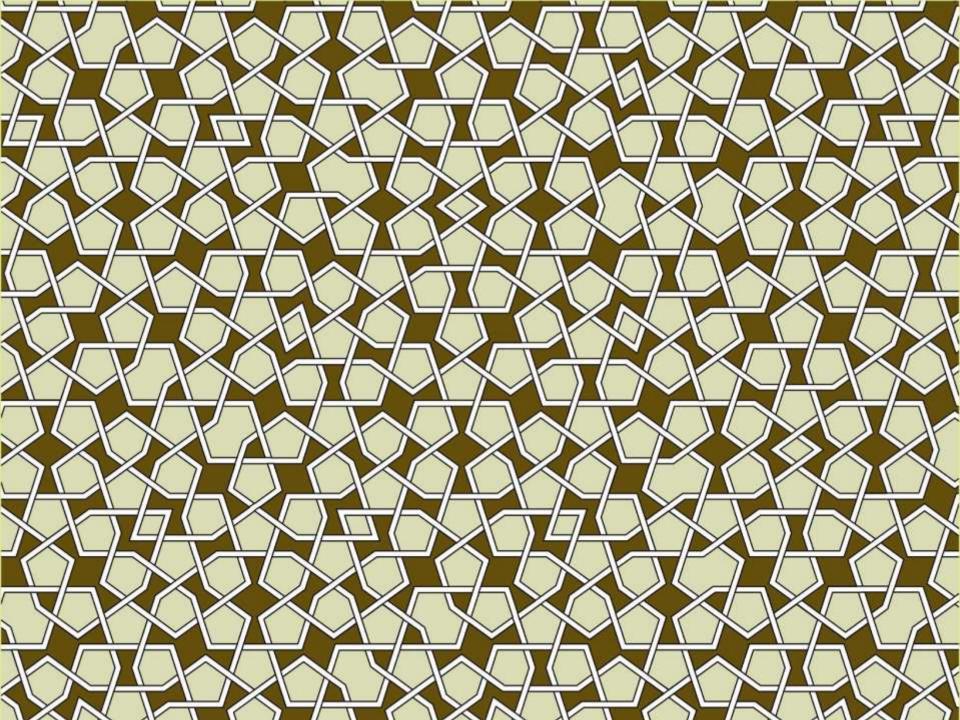


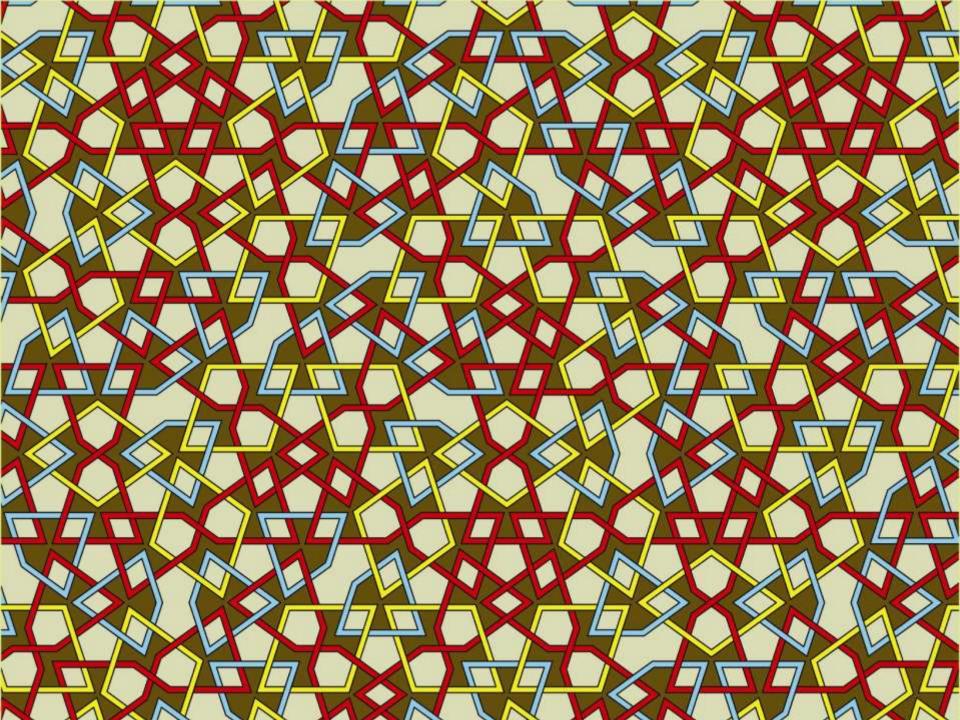


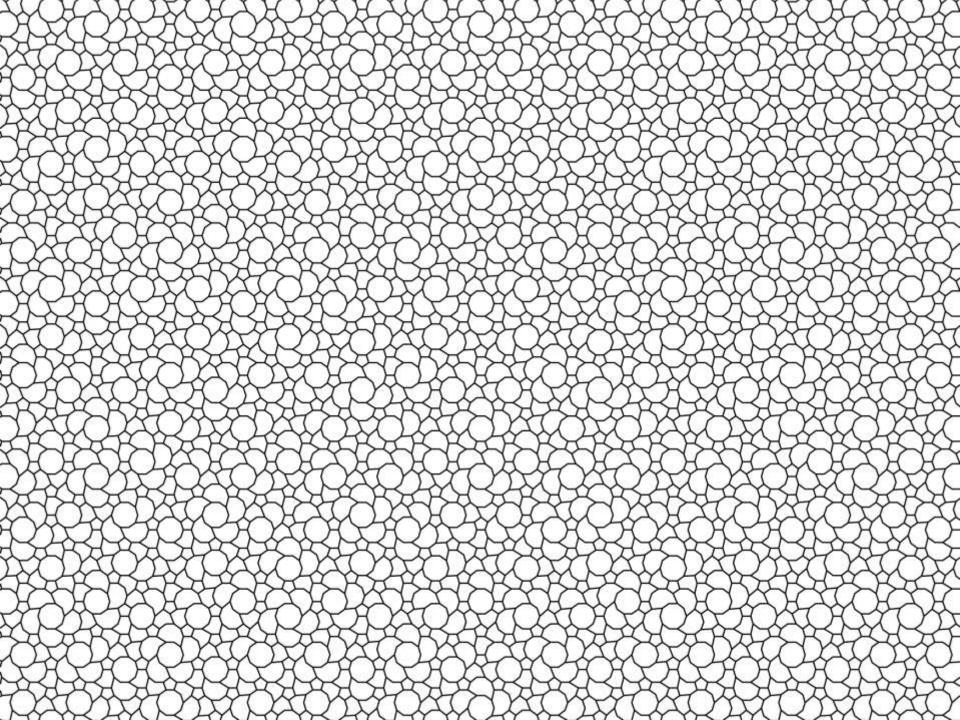


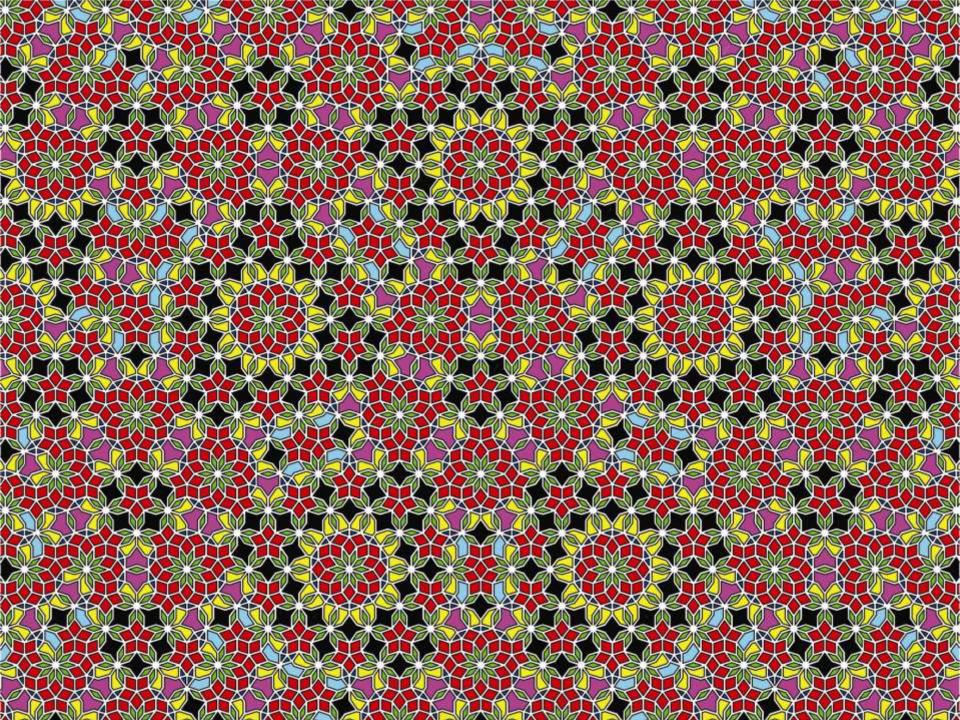


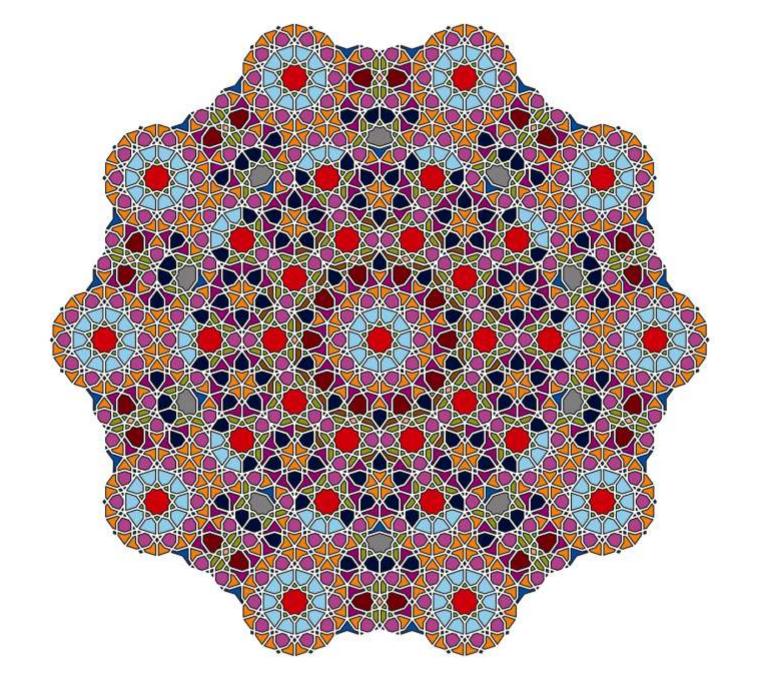


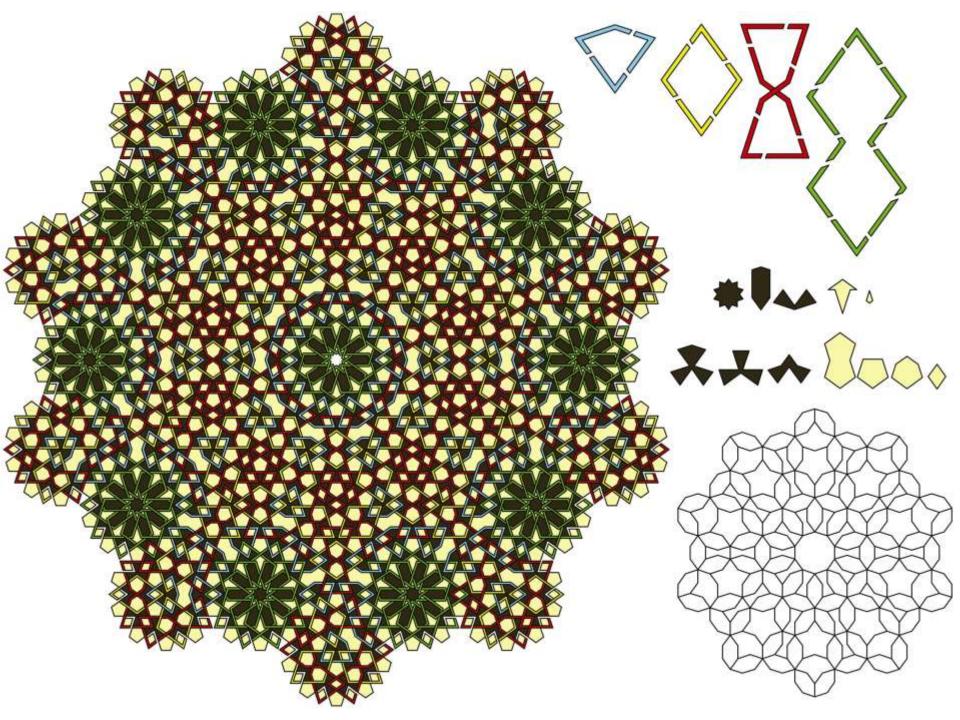


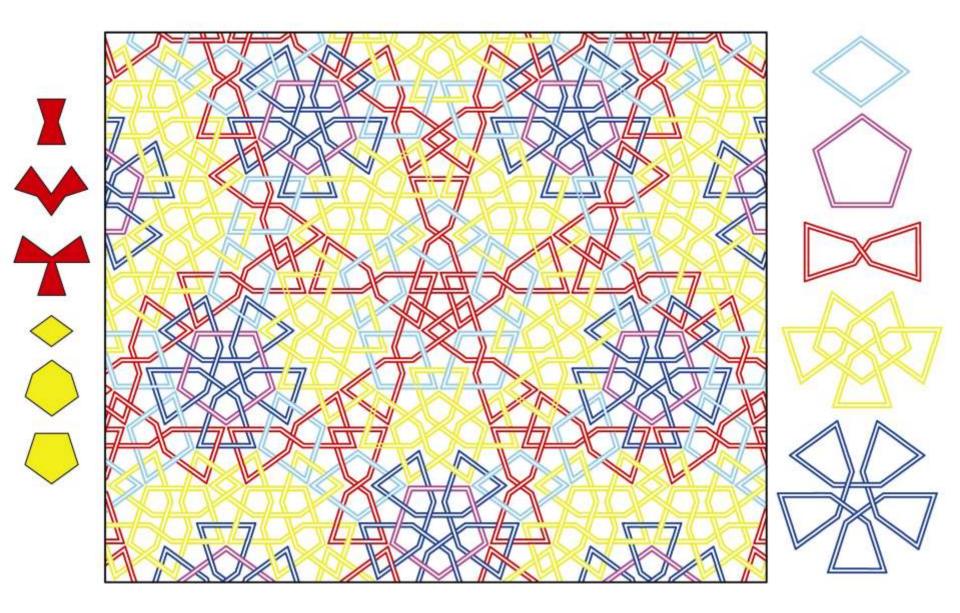


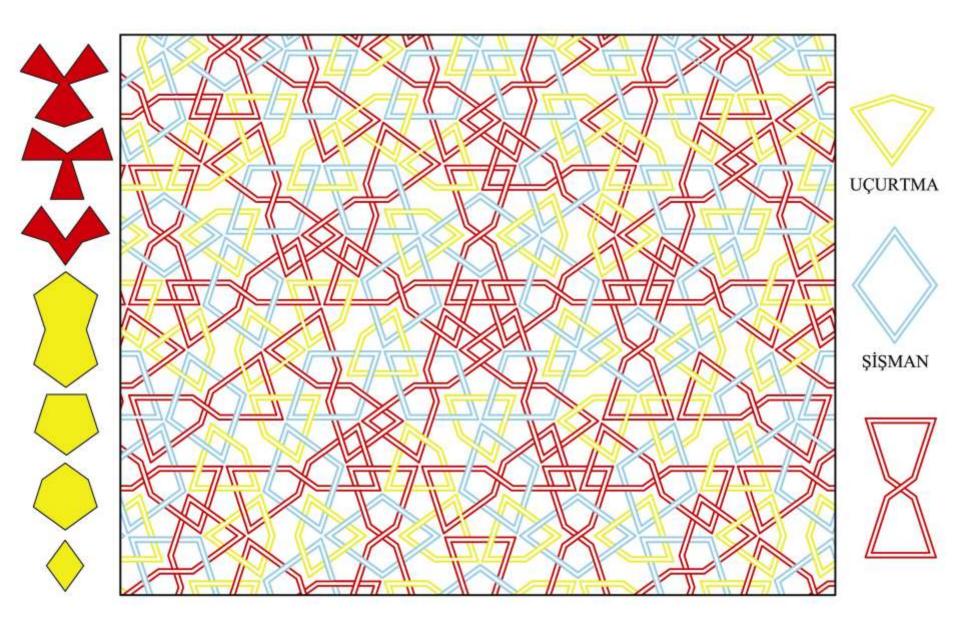












MODIFIED PENTAPLEX TILING

A pentaplex tiling which is modified by emptying certain decagonal regions and sometimes filling them with other decorations

